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A CONTROL VOLUME-BASED DISCRETIZATION OF THE REYNOLDS EQUATION FOR THE NUMERICAL SOLUTION OF ELASTOHYDRODYNAMIC LUBRICATION PROBLEMS

JAN ČERMÁK[1,*]

Department of Thermomechanics, Faculty of Mechanical Engineering, Technical University of Brno, Technická No. 2, Brno 616 69, Czech Republic

SUMMARY

This paper deals with the discretization of the one-dimensional Reynolds equation coupled with the film shape equation, that is used for the numerical solution of elastohydrodynamically lubricated contacts. The derivation of the developed discretization formula is based on the control volume approach. To reduce the discretization error caused by the upwind expression of the Couette (velocity) term, non-symmetric control volumes are used for discretization of the Reynolds equation, while for the elasticity equation the standard approach is used. A numerical method for the solution of the pressure and the film thickness profiles of elastohydrodynamically lubricated isothermal line contacts is presented. Results are presented for chosen typical parameters of a highly loaded contact. To show the formula efficiency, the convergence speed of both the presented discretization formula and a chosen comparative discretization formula (A.A. Lubrecht, *Ph.D. Thesis*, University of Twente, The Netherlands, 1987 and C.H. Venner, *Ph.D. Thesis*, University of Twente, The Netherlands, 1991) are checked. The results show that the presented formula gives better approximations of film thickness values for a given number of equidistant grid nodes. Moreover, the presented approach is probably suitable for more sophisticated cases, such as transient situations and elliptical contacts. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: elastohydrodynamic lubrication; line contact; Reynolds equation; numerical solution; control volume method

1. INTRODUCTION

Elastohydrodynamic lubrication (EHL or EHD lubrication) is the dominant regime of lubrication of highly loaded non-conform contacts [1]. Such contacts are used in a wide range of critical machine parts, e.g. roller bearings and gears. Reynolds distinguished that the flow of lubricant in a narrow gap between contact surfaces is of a creeping-flow nature [2] and derived the Reynolds differential equation in 1886. Only the Reynolds equation governs processes of a hydrodynamic lubrication nature and it is assumed that deformations of contact surfaces are negligible. In the case of EHL, the elastic deformations of surfaces have to be considered as they are non-negligible in comparison with the fluid film thickness. Therefore, it is necessary to solve the Reynolds equation coupled with the elasticity equation when dealing with EHL. Dowson and Higginson [3] were among the first who successfully solved this problem via a numerical simulation.

[1] E-mail: cermak@cis.vutbr.cz
* Correspondence to: ABB ENS, Olomoucka 7/9, Brno 656 66, Czech Republic.

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Many papers have been published on the numerical solution of EHL phenomena, i.e. on the determination of pressure, film thickness, temperature, etc. The problem is still unresolved and various authors prefer different types of procedures [1]. The numerical research has been concentrated on the line and point contact problems with different assumptions. However, the computations are very time consuming, in spite of the increasing computer power.

Whenever a digital computer is used to solve a differential equation, two problems arise. First there is the issue of discretization; this reduces the differential equation to a non-linear system of algebraic equations. The solution of the algebraic system is another matter. To reduce the computation time for large grids numerous research works have been performed and sophisticated numerical methods employed to solve the algebraic equations, e.g. the multilevel multigrid method [4,5].

An alternative way to obtain more accurate results and to reduce computation times, is to improve the discretization formula. It would be desirable to have a discretization formula so that coarser grids could be used to obtain a solution with the same discretization error, as for a standard discretization formula on a finer grid. The control volume method introduced by Patankar [6] is a modern method for discretization of differential equations, representing the laws of conservation exemplified by the Reynolds equation.

The scope of this paper is to use the control volume approach to obtain an efficient discretization formula for the EHL problem solution.

The aims of the paper are: (i) To introduce an improved discretization formula for the solution of the EHL line contact problem with the discretization based on the control volume philosophy. The formula attempts to reduce the error caused by the upwind discretization of the last term (Couette or velocity) of the Reynolds equation, while saving the numerical properties (smoothness of solutions, physically realistic solutions). (ii) To present the developed numerical method for the solution of discretization equations. (iii) To present results and perform comparisons of convergence speed for both a standard and the introduced discretization formulae.

The relatively simple case of the EHL line contact has been chosen as a benchmark, although all questions on the topic are not answered satisfactorily yet. However, the results clearly show the developed discretization formula's efficiency.

2. EQUATIONS

Equations used to solve the EHL line contact problem will be briefly introduced, as well as the constrains of the solution sought. (For further information see [7]). The one-dimensional Reynolds Equation (1) and fluid property Equations (3) and (4) describe the Newtonian fluid behavior. Note that Equation (3) is the so-called Roelands viscosity–pressure relationship and Equation (4) is the so-called Dowson and Higginson's density–pressure dependence.

The film shape (lubricant film thickness) at location x is described by Equation (2a,b). Considering the right-hand-side of Equation (2a), the first term represents the position of the first non-deformed contact surface relative to the second, the second term a separation due to geometry, and the third term an elastic deformation of both elastic surfaces. For the expression of the geometric separation of contact surfaces, a parabolic approximation of a circle is used. Materials of both contact solids have the same reduced modulus of elasticity E'. One of the solids is a plane and the other is of reduced radius R.

Steady state and smooth surfaces are assumed. Solids in the rolling contact are assumed to be infinitely long, thus no side leakage of the lubricant is expected. No thermal effects are assumed, which is only approximately true for pure rolling situations (adiabatic compression is neglected).

The pressure profile is constrained by the following conditions: (i) the pressure is equal to zero at the beginning of the contact region and (ii) the pressure is greater than or equal to zero in the whole contact region. A ventilation boundary condition determines the outlet point (the point at which the fluid film divides resulting in ventilation) [2], the Reynolds equation being no longer valid. Before this point is reached, the pressure profile reduces, becoming zero at this point. The constrain is realized by the computation algorithm by simply replacing a negative pressure value with zero.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\Gamma\frac{\mathrm{d}p}{\mathrm{d}x}\right)=12u\frac{\mathrm{d}(\rho h)}{\mathrm{d}x};\quad \Gamma=\frac{\rho h^3}{\eta}, \tag{1}$$

$$h(x)=h_0+\frac{x^2}{2R}+\delta(x), \tag{2a}$$

$$\delta(x)=2\frac{2}{\pi E'}\int_{-\infty}^{\infty}p(\xi)\ln|x-\xi|\,\mathrm{d}\xi, \tag{2b}$$

$$\eta=\eta_0\exp\left\{\frac{p_0\alpha}{z}\left[\left(1+\frac{p}{p_0}\right)^z-1\right]\right\}, \tag{3}$$

$$\rho=\rho_0\left(1+\frac{Ap}{1+Cp}\right). \tag{4}$$

3. DISCRETIZATION

Discretization is a process of replacing a differential equation (the solution of which is unknown) by a system of algebraic equations, the solution of which gives an approximation of the differential equation's solution over some discrete points (grid nodes). Some of the 'reasonable' requirements that a 'reasonable' discretization formula should satisfy have been formulated, e.g. see Reference [6]. However, only some of them are useful due to the integro-differential nature of the Reynolds equation coupled with the film shape equation.

For the development of the discretization formula, the control volume approach was chosen because of its simplicity and straightforwardness. The method is described by Patankar [6] with respect to applications in heat transfer and fluid flow. The fundamental idea is that the continuum is divided into control volumes and the laws of conservation are applicable to each. In such equations, the mass flow terms only appear at the control volume faces, and they need to be determined as accurately as possible. Consequently, it is necessary to express the mass flows via values of independent variables at the grid points. This results in an algebraic system. It can only be solved when the number of equations is equal to the number of unknowns, i.e. the control volume has to contain just one grid point.

In the present case, the Reynolds equation describes a mass flow conservation. For a control volume it is expressed by Equation (5), which can be obtained by the integration of the Reynolds equation over the control volume. The alternative way is to derive the mass conservation equation for the finite volumes directly, skipping the derivation of the differential equation. The control volume walls have been denoted as 'western' and 'eastern' by appropriate small letter subscripts in accordance with Patankar.

The problem now is to express the mass flow via a combination of grid point values of: pressure, film thickness, viscosity and density. The discretization formula (6a–g) has been developed for this purpose. The notation is explained in Figure 1. The terms of the mass conservation Equation (5) are described for the eastern wall only. The western wall terms are analogous. Note that the grid node is not placed in the middle of the control volume, but in the last three quarters. The design of such an unusual configuration was developed considering the fact that in the contact region, the Couette term of the Reynolds equation is several orders of magnitude higher than the Poiseuille one, due to the high viscosity values. The introduced non-symmetric first-order upwind discretization of the Couette (velocity) term (6a) expresses the term on the control volume wall more accurately, and causes a smaller error than a symmetric first-order upwind discretization. The Poiseuille term is expressed by the non-symmetric second-order discretization (6b,c). Unfortunately, the second-order symmetric discretization is more suitable than the non-symmetric discretization. However, it causes only a small error due to the small value of the term in comparison with the Couette one.

Altogether, the presented discretization formula, due to the dominance of the Couette term (being first-order), expresses the mass flow through the control volume wall better. A similar formula could be derived for the second-order one [8]; however, it is not guaranteed that convergence could be reached using the same numerical method and relaxation factors.

In the inlet region, the viscosity value is small and the above mentioned assumptions are do not hold. Thus, as a second step for future work, we recommend trying a discretization that depends on a ratio of the two terms.

When the deformation is evaluated (6e), it is supposed that the pressure is constant, not over the control volume, but over a symmetric domain (see Figure 2). The pressure value over the symmetric domain is equal to the grid node pressure. It can be said that 'the control volume for the Reynolds equation' and 'the control volume for the film shape equation' are 'shifted', the first one relative to the second one, while the grid node is the same for both control volumes.

$$\Gamma_{\mathrm{e}}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)_{\mathrm{e}} - \Gamma_{\mathrm{w}}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)_{\mathrm{w}} = 12u(\rho h)_{\mathrm{e}} - 12u(\rho h)_{\mathrm{w}}, \tag{5}$$

$$(\rho h)_{\mathrm{e}} = (\rho h)_{\mathrm{P}}, \tag{6a}$$

$$\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)_{\mathrm{e}} = \frac{p_{\mathrm{E}} - p_{\mathrm{P}}}{x_{\mathrm{E}} - x_{\mathrm{P}}}, \tag{6b}$$

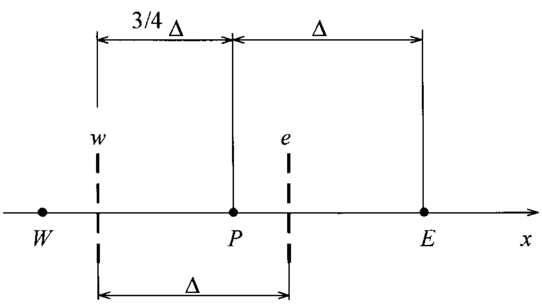

Figure 1. The control volume for discretization of the Reynolds equation. Walls are dashed and denoted by e (eastern) and w (western). Grid nodes are denoted by W, E and P.

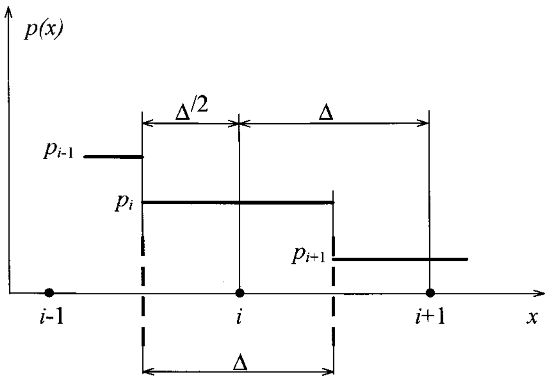

Figure 2. The symmetric domain for numerical computation of elastic deformations (boundaries are dashed). The pressure is assumed to be constant over the domains, and values are denoted p_{i-1}, p_i, and p_{i+1} in accordance with Equation (6e). Grid nodes are denoted $i-1$ (corresponding to W in Figure 1), i (corresponding to P), and $i+1$ (corresponding to E).

$$\Gamma_e = \frac{3}{4}\Gamma_P + \frac{1}{4}\Gamma_E, \tag{6c}$$

$$h_i = h_0 + \frac{x_i^2}{2R} + \delta_i, \tag{6d}$$

$$\delta_i = \sum_{j=1}^{n} p_j K_{i,j}, \tag{6e}$$

$$\text{for } i=j, \quad K_{i,j} = -2\frac{4}{\pi E'}\frac{\Delta}{2}\ln\left(\frac{\Delta}{2}\right); \tag{6f}$$

for $i \neq j$,

$$K_{i,j} = \frac{4}{\pi E'}\left[\left(|x_i - x_j| - \frac{\Delta}{2}\right)\ln\left(|x_i - x_j| - \frac{\Delta}{2}\right) - \left(|x_i - x_j| + \frac{\Delta}{2}\right)\ln\left(|x_i - x_j| + \frac{\Delta}{2}\right)\right]. \tag{6g}$$

4. NUMERICAL METHOD

To obtain an approximate solution of our integro-differential Equation (1), we have to solve the non-linear discretization Equations (5) and (6a–g) for each grid point. An iterative numerical scheme is commonly used for this purpose. The Gauss Seidel's underrelaxation iterative method was chosen. To obtain a new approximation, the one-dimensional method of cuttings has been employed. The method of cuttings is suitable for testing discretization formulae because derivatives are computed numerically. However, more computation is required.

Note that Lubrecht [9] and Venner [10] used a similar numerical method for their multigrid solver, but they employed the one-dimensional Newton method instead of the method of cuttings. However, they did not express all derivatives, but only a part of the more important ones, while the rest were neglected.

It would not be appropriate to explain all the algorithm details here. The program source text (including comments) in language C is available from the author.

5. RESULTS AND COMPARISON

For the presentation of results and for comparisons of the introduced discretization formula with previously used ones, a set of typical parameters for a highly loaded steel-to-steel line contact, flooded with a mineral oil, is chosen and introduced in Table I. There have been many series of dimensionless parameters introduced by a number of authors. In Table I, the Dowson and Higginson's dimensionless parameters are used. An equidistant grid is used for all computations. The pressure profile and film thickness for the number of nodes, $n = 1000$, are depicted in Figures 3 and 4.

The second (and main) task of this contribution is to compare the presented discretization formula with another widely used one, from the convergence speed point of view. The formula that was successfully used by Lubrecht [9] and Venner [10], has been chosen as the comparative one. It is introduced by Equation (7a–c). The notation is the same as the previously used one. Elastic deformations are computed the same way. Note that this comparative formula differs from the presented one in the grid node placement only. The Poiseuille (pressure) term (7b,c) is discretized via the second-order central discretization and the Couette (velocity) term (7a) via the first-order upwind discretization. The grid node is in the middle of the control volume if we investigate the comparative formula from the control volume approach point of view. No 'shifted control volumes' are used. The formula is also first-order with regard to the Couette term dominance.

The comparison has been performed by investigating the results for both discretization formulae and different number of nodes: $n = 100$, 200, 300, 400, 600, 700, 800, 900 and 1000. The convergence of important parameters of the solution have been compared: the minimum film thickness value and the central film thickness value (the film thickness at $x = 0$ location).

Table I. Chosen typical parameters of a highly loaded EHL steel-to-steel line contact flooded by a mineral oil

Parameter	Description	Value
u	Speed of contact surfaces (mean value)	$5.942 \cdot 10^{-1}$ m s^{-1}
E'	Reduced modulus of elasticity	$2.198 \cdot 10^{11}$ Pa
R	Reduced radius of non-deformed contact surfaces	$1.11130 \cdot 10^{-2}$ m
w	Load per unit length	$1.225 \cdot 10^{5}$ kg m^{-1}
b	Half-width of dry (Hertzian) contact	$1.256 \cdot 10^{-4}$
z	Constant in Equation (3)	0.6960
p_0	Constant in Equation (3)	$1.980 \cdot 10^{8}$ Pa
α	Pressure–viscosity exponent	$2.276 \cdot 10^{-8}$ Pa^{-1}
η_0	Viscosity at atmospheric pressure	$4.110 \cdot 10^{-2}$ Pa s
A	Constant in Equation (4)	$5.820 \cdot 10^{-11}$
C	Constant in Equation (4)	$1.680 \cdot 10^{-10}$
ρ_0	Density at atmospheric pressure	$8.66 \cdot 10^{2}$ kg m^{-3}
x_{in}	Start of the considered contact region	$-4.400 \cdot 10^{-4}$ m
x_{out}	End of the considered contact region	$1.700 \cdot 10^{-4}$ m
U	Dimensionless speed parameter $\left(= \dfrac{\eta_0 u}{E'R} \right)$	$1.000 \cdot 10^{-11}$
G	Dimensionless material parameter $(= \alpha E')$	5002
W	Dimensionless load parameter $\left(= \dfrac{w}{E'R} \right)$	$5.005 \cdot 10^{-5}$

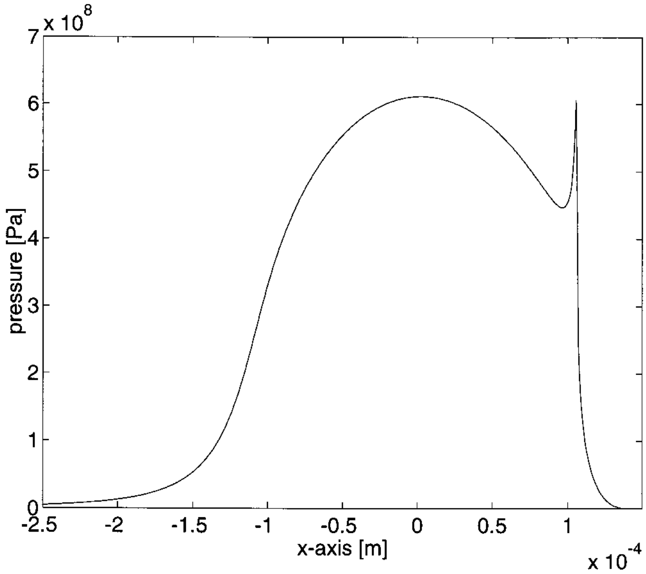

Figure 3. Pressure profile of the isothermal EHL line contact. Parameters are listed in Table I. Surfaces are rolling from left to the right, thus the inlet region is on the left.

The secondary pressure maximum value does not show any reasonable trend and was not investigated. Results of the investigation are introduced in graphic form in Figure 5 (the minimum film thickness) and Figure 6 (the central film thickness). The accurate values from which Figures 5 and 6 were constructed, are introduced in Table II.

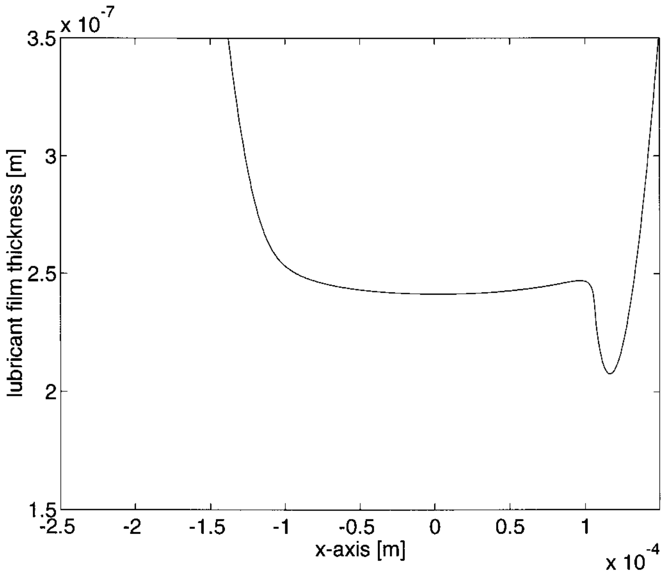

Figure 4. Film thickness profile of the isothermal EHL line contact. Parameters are listed in Table I. Surfaces are rolling from left to the right, thus the inlet region is on the left.

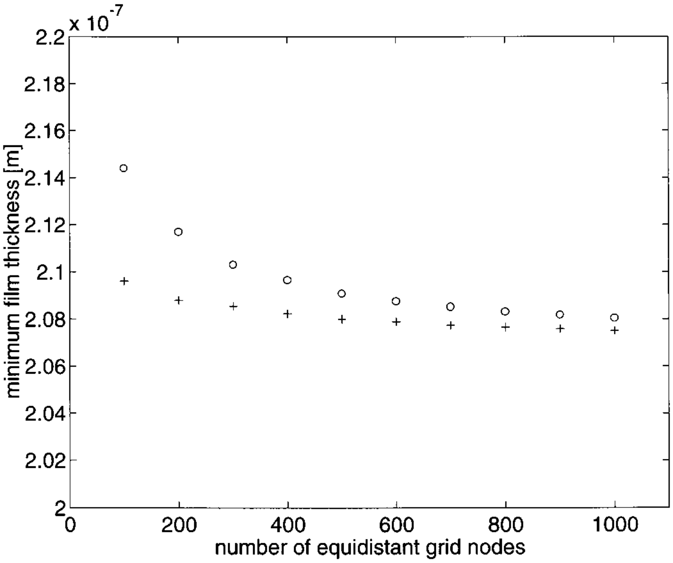

Figure 5. Minimum film thickness values for different numbers of equidistant grid nodes. The presented formula results are denoted by +, while the comparative results are denoted by ○.

It is clear from both the figures and the table that the presented discretization formula (6a–g) gives more accurate values for the central and minimum film thickness compared with Equation (7a–c), while benefits of the first-order discretization, such as convergence behavior, are kept. To reach convergence, the same relaxation factors were used for both the presented and comparative formulae. Moreover, the presented formula gives an underestimation of the

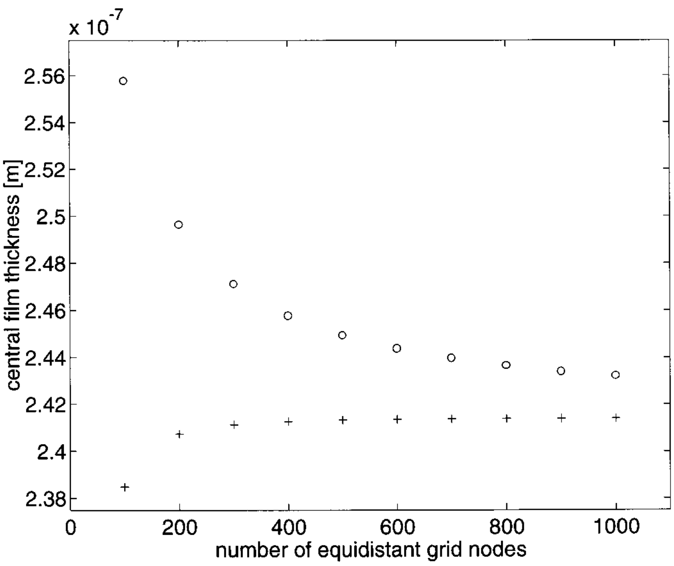

Figure 6. Central film thickness values for different numbers of equidistant grid nodes. The presented formula results are denoted by +, while the comparative results are denoted by ○.

Table II. Central and minimum film thickness values computed for different numbers of grid nodes using the presented discretization formula and the comparative (Lubrecht and Venner) formula[a]

Number of grid nodes	Minimum film thickness ($\cdot 10^{-7}$ m)		Central film thickness ($\cdot 10^{-7}$ m)	
	Presented formula	Lubrecht and Venner's formula	Presented formula	Lubrecht and Venner's formula
100	2.0961	2.1439	2.3848	2.5577
200	2.0879	2.1169	2.4073	2.4963
300	2.0854	2.1030	2.4113	2.4712
400	2.0823	2.0965	2.4127	2.4577
500	2.0800	2.0907	2.4133	2.4494
600	2.0789	2.0875	2.41358	2.4437
700	2.0774	2.0851	2.41374	2.4396
800	2.0766	2.0831	2.41385	2.4365
900	2.0759	2.0817	2.41390	2.4340
1000	2.0751	2.0804	2.41395	2.4321

[a] The same values are depicted in Figures 5 and 6.

central film thickness and an overestimation of the minimum film thickness, while the comparative formula gives an overestimation of both.

Both methods require approximately the same number of iterations to reduce values of the algebraic system residues below a given value. The value was chosen to be very low, (10^{-10}) for all computations, to ensure that the solution is not influenced by this kind of error. Thus, a double precision type of 16 significant digits was used.

$$(\rho h)_{\mathrm{e}} = (\rho h)_{\mathrm{P}}, \tag{7a}$$

$$\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)_{\mathrm{e}} = \frac{p_{\mathrm{E}} - p_{\mathrm{P}}}{x_{\mathrm{E}} - x_{\mathrm{P}}}, \tag{7b}$$

$$\Gamma_{\mathrm{e}} = \frac{1}{2}\Gamma_{\mathrm{P}} + \frac{1}{2}\Gamma_{\mathrm{E}}. \tag{7c}$$

6. CONCLUSION

A solution method for the isothermal, steady state EHL line contact problem was introduced. The basis of this methodology is the developed first-order discretization formula. This is more accurate than the previously used one (by Lubrecht [9] and Venner [10]), proved by the comparison of results given by both formulae. Recommendations for further research has been outlined. Although the benefit of the formula has been shown on the isothermal line contact case under steady state conditions, it needs to be used in more complicated situations, such as transient situations and elliptical contacts. Research in those fields has just started.

APPENDIX A. NOMENCLATURE

A 1 lubricant density coefficient

C	1	lubricant density coefficient
e		(eastern) control volume wall between grid points P and E
E		grid point left of the control volume
E'	Pa	reduced modulus of elasticity
h	m	film thickness
h_0	m	constant part of the film thickness
i		grid point
p	Pa	pressure
p_0	Pa	Roeland's parameter of lubricant
P		grid point within the control volume
R	m	reduced radius of contact surfaces
u	m s^{-1}	mean velocity of surfaces $(u_1+u_2)/2$
w		(western) control volume wall between grid points P and W
w	N m^{-1}	line contact load per unit length
W		grid point right of the control volume
x	m	axis in the direction of rolling
z	1	Roeland's parameter of lubricant
Δ	m	distance between grid nodes and control volume length
Γ	m s	'diffusion' coefficient in the Reynolds equation
α	Pa^{-1}	pressure–viscosity coefficient
δ	m	elastic deformation of both contact surfaces
η_0	Pa s	viscosity at atmospheric pressure
η	Pa s	viscosity at a given pressure
ρ_0	kg m^{-3}	density at atmospheric pressure
ρ	kg m^{-3}	density at a given pressure

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